


① Prove $n - e + f = 2$ using induction on e

$$\Rightarrow \begin{matrix} n=1 \\ e=1 \\ f=2 \end{matrix} \Rightarrow 1 - 1 + 2 = 2 \checkmark$$

Base: $P(1)$  $\Rightarrow \begin{matrix} n=2 \\ e=1 \\ f=1 \end{matrix} \Rightarrow 2 - 1 + 1 = 2 \checkmark$

I.H.: Assume for some $P(k)$ $k > 1$ that the formula holds, $P(k)$ connected

I.S.: Consider graph $G = P(n)$ $n > k$

Case 1: G has no cycles $\Rightarrow G$ is a tree

$$n = n$$

$$e = n - 1 \Rightarrow n - (n - 1) + 1 = 2 \checkmark$$

$$f = 1$$

Case 2: G has at least one cycle

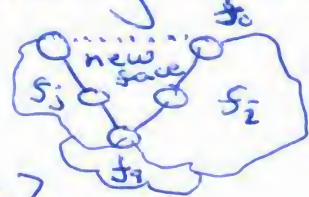
- Select e from a cycle

- $H = G - e$, I.H. on H

- So $n_H + e_H + f_H = 2$ holds

- Add back the edge to e

- We note that creating a cycle on a planar embedding also creates a face



$$n_G = n_H$$

$$e_G = e_H + 1 \Rightarrow n_H + e_H + f_H = 2$$

$$f_G = f_H + 1$$

$$n_G - (e_G - 1) + (f_G - 1) = 2$$

$$n_G - e_G + f_G = 2 \quad \square$$

② G minimal non-planar

$H = G - e$ for some $e \in E(G)$

$\exists e$ s.t. H is maximal planar?

- Removing any edge from G makes the resulting graph planar
- The smallest such nonplanar graphs must necessarily be a K_5 or $K_{3,3}$ subdivision
- For H to be maximally planar, adding any single edge needs to create a K_5 or $K_{3,3}$ subdivision
- On a planar embedding of H , we can naively add edges from a vertex v on a subdivided edge to neighbors of its 'hub' without crossing by following a ~~route~~ back to the hub then to a neighbor



- Hence, G must not have any subdivided edges

\Rightarrow Only holds when

$G = K_5$ ~~or $K_{3,3}$~~ \square

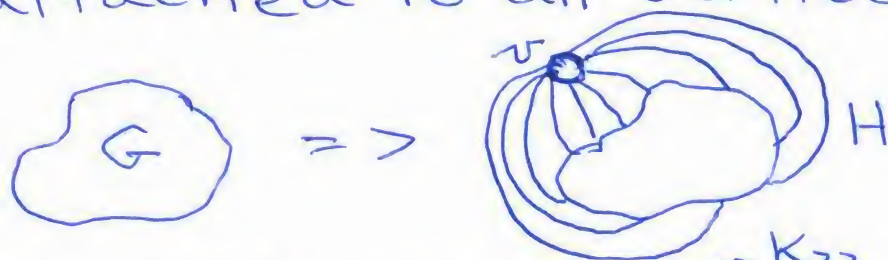
- On $K_{3,3}$ can still add edges within set

$K_5 - e$ subdivision

③ G is outer planar iff it contains no K_4 or $K_{2,3}$ subdivision

(\Rightarrow)-Let G be outer planar

- Define $H = G + v$, where v is placed in the outer face of G and attached to all vertices in G



- This cannot create K_5 or $K_{3,3}$ subdivision, as H is planar \Rightarrow so G has no K_4 or $K_{2,3}$ subdivision \checkmark

(\Leftarrow) - G has no K_4 or $K_{2,3}$ subdivision and is therefore at least planar

- Construct $H = G + v$ as before, H is also planar, as it has no K_5 or $K_{3,3}$ sub.
 - From our proof of Kuratowski's theorem, we know \exists an embedding with v on outer face
 - We note that v can reach all $u \in V(G)$ without any edges crossing
 - Hence, all $u \in V(G)$ must be on the outer face of an embedding of G
- $\Rightarrow G$ is outer planar \square

④ Show k -regular graph with cut vertex v must have $\chi'(G) > k$

- We consider that $\chi'(G) = k$
- We note that each edge color will form a matching on G
- As $\chi'(G) = d(v) \forall v \in V(G)$ each vertex is an endpoint on all k matchings
 \Rightarrow Each edge color forms a perfect match
- Consider H_1 as a component of $G - v$
- We note from $u \in V(H_1)$ there are some $x < k$ vertices in $N(v)$, hence there exists at least one color with a perfect match on $H_1 \Rightarrow |V(H_1)| = \text{even}$
- We make the same argument on the other component of $G - v \Rightarrow H_2$
- So $G - v$ as well as G itself all have perfect matches
 \Rightarrow contradiction, as the parity of the ~~edge~~ set can't be even for both \square

⑤ Prove for what n on a $4 \times n$ chess board a knight's tour is possible

- We note a "knight's tour" is just a tricky way to say "Hamiltonian cycle" on graph constructed from possible movements on the board
- Recall that $c(G-S) \leq |S| \forall S \subseteq V(G)$ for a Hamiltonian cycle to exist
- See how we define S below

	x		x		x	
	S		S		S	
S		S		S		
x		x		x		

- Note that each x will be disconnected from $G-S$ as a single vertex
- We have $|S|$ 'x' components plus at least one larger component, so

$$c(G-S) = |S| + 1 \leq |S| \text{ doesn't hold}$$

\Rightarrow No such n allows a tour \square